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# Notes on submanifolds in warped products

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ABSTRACT

In this paper, we study submanifolds in the warped product $M \times_f \mathbb{R}$, where $M$ is a Riemannian manifold, $f$ is the warping function. We investigate the graphs with positive Gaussian curvature in $M^2 \times \mathbb{R}$, the minimal graphs in $M \times_f \mathbb{R}$ and the totally geodesic submanifolds in $M \times_f \mathbb{R}$.



## 1. Introduction

A lot of research about submanifolds in product manifolds has been done in the last years (see [1–7,13,15,16]). In [1–3,15] the authors studied the submanifolds in the products $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$. In [4–7,13,16], the ambient spaces of submanifolds are extended to general product manifolds or warped product manifolds. Many important results have been obtained in this research area.

In this paper, we consider submanifolds in the warped product $\overline{M} = M \times_f \mathbb{R}$, where $M$ is a Riemannian manifold, $\mathbb{R}$ is the 1-dimensional Euclidean space with the standard metric, and $f$ is the warping function. We study the Gauss curvature of a graph in $M^2 \times \mathbb{R}$, the minimal graphs in $M \times_f \mathbb{R}$ and the totally geodesic submanifolds in $M \times_f \mathbb{R}$. The main results of this paper are Theorem 3.1, Theorem 4.1, Theorem 5.2 and Theorem 5.3.

The rest of this paper is organized as follows. In Section 2 we recall some notations and known results about warped products, graphs in a warped product, and the pole and strongly symmetry of a Riemannian manifold. In Section 3 we investigate graphs with positive Gaussian curvature in $M^2 \times \mathbb{R}$. Section 4 is devoted to the study of radially symmetric minimal graphs in the warped product $M \times_f \mathbb{R}$. In Section 5 we give some interesting properties of totally geodesic submanifolds in $M \times_f \mathbb{R}$.

## 2. Preliminaries

Throughout this paper, all manifolds are assumed to be smooth connected manifolds without boundary. If $N$ is a manifold and $q \in N$, $T_qN$ or $N_q$ will denote the tangent space to $N$ at $q$.

First we recall some notations about warped product (see [14]). Let $M$ be an $n$-dimensional Riemannian manifold with metric tensor $g$, $\mathbb{R}$ be the 1-dimensional Euclidean space with its standard metric $\tilde{g}$, and $f$ is a positive smooth function defined on $M$. According to definition, the warped product $\overline{M} = M \times_f \mathbb{R}$ is the product manifold $M \times \mathbb{R}$ furnished with the Riemannian metric

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$$\bar{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\tilde{g}),$$

where $\pi : M \times \mathbb{R} \to M$ and $\sigma : M \times \mathbb{R} \to \mathbb{R}$ denote the projections of the product. $f$ is called the warping function of the warped product. If the warping function $f = 1$ in $\overline{M} = M \times_f \mathbb{R}$, then $\overline{M} = M \times \mathbb{R}$ is simply the Riemannian product. For $x \in M$ and $t \in \mathbb{R}$, $\pi^{-1}(x) = \{x\} \times \mathbb{R}$ are called the fibers and $\sigma^{-1}(t) = M \times \{t\}$ are called the leaves. It is a known fact that all leaves are totally geodesic. Vectors of $\overline{M}$ tangent to leaves are called to be horizontal and tangent to fibers are called to be vertical. This yields the direct decomposition

$$T_{(x,t)}(\overline{M}) = T_{(x,t)}\big(M \times \{t\}\big) \oplus T_{(x,t)}\big(\{x\} \times \mathbb{R}\big).$$

Let $\mathscr{H} : T_{(x,t)}(\overline{M}) \to T_{(x,t)}(M \times \{t\})$ and $\mathscr{V} : T_{(x,t)}(\overline{M}) \to T_{(x,t)}(\{x\} \times \mathbb{R})$ be the horizontal projection and the vertical projection, respectively.

Let $\overline{M} = M \times_f \mathbb{R}$, $\Omega$ be an open subset of $M$, and $\varphi \in C^\infty(\Omega)$. The subset of $\overline{M}$

$$\Sigma = \big\{\big(x, \varphi(x)\big) \bigm| x \in \Omega\big\}$$

is called the graph of the function $\varphi$. The graph $\Sigma$ is said to be entire if $\Omega = M$. Define

$$w(x,t) = t - \varphi(x), \quad (x,t) \in \Omega \times \mathbb{R} \subset \overline{M}. \tag{1}$$

Then we have $\Sigma = w^{-1}(0)$. Denote by $\partial_t$ the coordinate vector field of $\mathbb{R}$. We also use the same symbol $\partial_t$ to denote the lift of the coordinate vector field of $\mathbb{R}$ to $\overline{M}$ (see [14]). Let $\overline{\nabla}$ and $\nabla$ denote the gradients on $\overline{M}$ and $M$ respectively. A computation shows that

$$\overline{\nabla} w = \frac{1}{f^2}\partial_t - \nabla\varphi,$$

where $\nabla\varphi$ and $\partial_t$ are understood as a vector field on $\overline{M}$ by lift to $\overline{M}$. Define

$$\xi \equiv \frac{\overline{\nabla} w}{\|\overline{\nabla} w\|_{\overline{M}}} = \frac{\partial_t - f^2\nabla\varphi}{\|\partial_t - f^2\nabla\varphi\|_{\overline{M}}}. \tag{2}$$

$\xi|_\Sigma$ is a unit normal vector field of $\Sigma$.

Let $II(\cdot,\cdot)$ be the vector valued second fundamental form of $\Sigma$. Define the real valued second fundamental form $l$ of $\Sigma$ by

$$II(X,Y) = -l(X,Y)\xi \tag{3}$$

for any tangent vector fields $X, Y$ of $\Sigma$. The Gaussian curvature $K$ of $\Sigma$ at a point $p \in \Sigma$ is

$$K(p) = \frac{l(u,u)l(v,v) - l(u,v)^2}{\bar{g}(u,u)\bar{g}(v,v) - \bar{g}(u,v)^2}, \tag{4}$$

where $u, v \in T_p\Sigma$ is linear independent (see [8, p. 186]). The mean curvature vector field $\vec{H}$ of $\Sigma$ is defined by $\vec{H} = \frac{1}{n}\sum_{i=1}^n II(e_i, e_i)$, where $e_1, \dots, e_n$ is a local field of orthonormal frames of $\Sigma$. $H \equiv \bar{g}(\vec{H}, \xi)$ is called the mean curvature of $\Sigma$ with respect to $\xi$.

Then we recall some notations about the pole and strongly symmetry of a Riemannian manifold (see [9,17]). A point $o$ in a Riemannian manifold $N$ is called a pole of $N$ iff the exponential mapping $\exp_o$: $N_o \to N$ is a diffeomorphism. We refer to $(N, o)$ as a manifold $N$ with a pole $o$. A manifold with a pole $(N, o)$ is strongly symmetric around $o$ iff every linear isometry $\phi : N_o \to N_o$ is realized as the differential of an isometry $\Phi : N \to N$, i.e., $\Phi(o) = o$ and $\Phi_*|_o = \phi$, where $\Phi_*|_o$ denotes the differential of $\Phi$ at $o$. (In [9] the authors use the term "model" for "strongly symmetric manifold".)

Let $(N, o)$ be strongly symmetric around $o$. For any $x \in N$, denote by $r = r(x)$ the geodesic distance from $o$ to $x$. If $h$ is a function defined on an interval $[0, R)$, then $h(r(x))$ is a function defined on the open ball $B(o, R)$ in $N$. We call a function $h$ with form $h = h(r(x))$ to be radially symmetric around $o$. If $h = h(r(x))$ is a smooth radially symmetric function, then a simple computation shows that

$$\Delta_N h = h'' + (\Delta_N r)h', \tag{5}$$

where $\Delta_N$ denotes the Laplacian on $N$ (see [9,17]).

## 3. Graphs with positive Gaussian curvature in $M^2 \times \mathbb{R}$

In this section we investigate graphs in simple Riemannian product $\overline{M} = M^2 \times \mathbb{R}$, where $M^2$ is a 2-dimensional flat Riemannian manifold.

**Theorem 3.1.** *Let $M$ be a 2-dimensional flat connected complete Riemannian manifold. Consider the Riemannian product $\overline{M} = M \times \mathbb{R}$, and the entire graph $\Sigma = \{(x, \varphi(x)) \mid x \in M\}$ in $\overline{M}$ defined by $\varphi \in C^\infty(M)$. Denote by $K$ the Gaussian curvature of $\Sigma$. If $K$ is positive, that is, for every $p \in \Sigma$, $K(p) > 0$, then $\varphi$ is unbounded. Consequently, if $M$ is compact, then there is no entire graph with positive Gaussian curvature in $\overline{M}$.*

**Proof.** Since $M$ is flat, $M$ is locally isometric to $\mathbb{R}^2$. Thus, for any point $p \in M$, we can choose a locally coordinate neighborhood $U$ near $p$ such that the coordinate vector fields $\partial_1 \equiv \frac{\partial}{\partial x^1}$, $\partial_2 \equiv \frac{\partial}{\partial x^2}$ form a local orthonormal frame ($(x^1, x^2)$ denotes the local coordinates). Consider the map

$$\Phi : M \to \Sigma \subset \overline{M}, \quad x \mapsto \big(x, \varphi(x)\big).$$

Let $u_i = d\Phi(\partial_i)$, $i = 1, 2$. Then we have

$$K = \frac{l(u_1, u_1)l(u_2, u_2) - l(u_1, u_2)^2}{\bar{g}(u_1, u_1)\bar{g}(u_2, u_2) - \bar{g}(u_1, u_2)^2}$$

on $\Phi(U) \subset \Sigma$. Note that $\bar{g}(u_1, u_1)\bar{g}(u_2, u_2) - \bar{g}(u_1, u_2)^2 > 0$, then $K(p) > 0$ is equivalent to

$$l(u_1, u_1)l(u_2, u_2) - l(u_1, u_2)^2 > 0. \tag{6}$$

Introducing coordinates $(x^1, x^2, t)$ on $U \times \mathbb{R}$ and setting $\partial_t = \frac{\partial}{\partial t}$, a computation shows that

$$u_i = d\Phi(\partial_i) = \partial_i + \frac{\partial \varphi}{\partial x^i}\partial_t, \quad i = 1, 2.$$

Let $\overline{\nabla}$ and $\nabla$ be the Riemannian connections of $\overline{M}$ and $M$ respectively. We have

$$\begin{aligned}
l(u_i, u_j) &= -\bar{g}\big(II(u_i, u_j), \xi\big) \\
&= -\bar{g}\left(II\left(\partial_i + \frac{\partial \varphi}{\partial x^i}\partial_t, \partial_j + \frac{\partial \varphi}{\partial x^j}\partial_t\right), \xi\right) \\
&= -\bar{g}\left(\overline{\nabla}_{\partial_i + \frac{\partial \varphi}{\partial x^i}\partial_t}\left(\partial_j + \frac{\partial \varphi}{\partial x^j}\partial_t\right), \xi\right) \\
&= -\bar{g}\left(\overline{\nabla}_{\partial_i}\partial_j + \overline{\nabla}_{\partial_i}\frac{\partial \varphi}{\partial x^j}\partial_t + \frac{\partial \varphi}{\partial x^i}\overline{\nabla}_{\partial_t}\partial_j + \frac{\partial \varphi}{\partial x^i}\overline{\nabla}_{\partial_t}\frac{\partial \varphi}{\partial x^j}\partial_t, \xi\right).
\end{aligned}$$

Note that $\overline{\nabla}_{\partial_i}\partial_j = \nabla_{\partial_i}\partial_j$ (understood as lift to $\overline{M}$), $\overline{\nabla}_{\partial_i}\partial_t = \overline{\nabla}_{\partial_t}\partial_j = \overline{\nabla}_{\partial_t}\partial_t = 0$ (see [14, p. 206, Proposition 35]). Thus, by use of (2), we get

$$l(u_i, u_j) = -\frac{\partial^2 \varphi}{\partial x^i \partial x^j}\bar{g}(\partial_t, \xi) = -\frac{1}{\psi}\frac{\partial^2 \varphi}{\partial x^i \partial x^j}, \tag{7}$$

where $\psi = \|\partial_t - \nabla\varphi\|_{\overline{M}}$. Hence we obtain

$$l(u_1, u_1)l(u_2, u_2) - l(u_1, u_2)^2 = \frac{1}{\psi^2}\det\left(\frac{\partial^2 \varphi}{\partial x^i \partial x^j}\right). \tag{8}$$

If $K > 0$, then $\det(\frac{\partial^2 \varphi}{\partial x^i \partial x^j}) > 0$, so we have

$$\frac{\partial^2 \varphi}{(\partial x^1)^2}\frac{\partial^2 \varphi}{(\partial x^2)^2} > \left(\frac{\partial^2 \varphi}{\partial x^1 \partial x^2}\right)^2 \geqslant 0.$$

Therefore, at any point of $M$, we have

$$\Delta\varphi = \frac{\partial^2 \varphi}{(\partial x^1)^2} + \frac{\partial^2 \varphi}{(\partial x^2)^2} \neq 0,$$

where $\Delta\varphi$ denotes the Laplacian of $\varphi$ on $M$. Since $M$ is connected, $\Delta\varphi$ is $> 0$ or $< 0$ on $M$, that is, $\varphi$ is subharmonic or superharmonic on $M$. By Hopf's lemma, it is obvious that $M$ is necessarily noncompact. Because $M$ is flat, complete and with dimension 2, by a result from Huber and Karp (see [12, Corollary B], or see [10,11]), we see that $M$ is strongly parabolic, that is, it admits no nonconstant negative subharmonic function. If $\varphi$ is bounded, then $\varphi$ is constant and hence $\Delta\varphi = 0$, this is a contradiction. Therefore $\varphi$ is unbounded.

The rest of the theorem is obvious since any smooth function $\varphi$ on $M$ must be bounded by the compactness of $M$. □

## 4. Radially symmetric graphs in $M \times_f \mathbb{R}$

In this section we investigate graphs in the warped product $\overline{M} = M \times_f \mathbb{R}$, where $M$ is an $n$-dimensional Riemannian manifold with a pole $o$ and is strongly symmetric around $o$.

**Theorem 4.1.** *Let $M$ be an $n$-dimensional Riemannian manifold with a pole $o$ and suppose that $M$ is strongly symmetric around $o$. Consider the warped product $\overline{M} = M \times_f \mathbb{R}$. Let $\Omega = B(o, R)$ be the open geodesic ball in $M$ with center $o$ and radius $R$. Let $\varphi \in C^\infty(\Omega)$ and $\Sigma = \{(x, \varphi(x)) \mid x \in \Omega\}$ be the graph in $\overline{M}$ defined by $\varphi$. Suppose that both $f$ and $\varphi$ are radially symmetric around $o$, $f(r)$ is increasing with respect to $r$ ($r = r(x)$ denotes the geodesic distance from $o$ to $x$), and that $\Sigma$ is a minimal submanifold of $\overline{M}$. Let $S_r$ be the geodesic sphere of $M$ with center $o$ and radius $r$ and $V(r)$ be the volume of $S_r$. Then*:

(1) *If $V(r)f(r)$ is increasing on $[0, R)$, in particular if $V(r)$ is increasing on $[0, R)$, then $\varphi$ is constant on $\Omega$, that is, $\Sigma$ is a piece of leave of the warped product.*
(2) *There exists $R_0 \in (0, R)$ such that $\varphi$ is constant on $B(o, R_0)$.*
(3) *If $M$ is a Cartan–Hadamard manifold, that is, a complete simply-connected Riemannian manifold of nonpositive sectional curvature, then $\varphi$ is constant on $\Omega$.*

**Proof.** We first compute the mean curvature $H$ of $\Sigma$. Choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{n+1}$ of $\Omega \times \mathbb{R}$ such that $e_{n+1} = \xi$. It is obvious that $e_1, \ldots, e_n$ are tangent to $\Sigma$ when restricted to $\Sigma$. According to definition of the mean curvature,

$$\begin{aligned} nH &= \bar{g}\left(\sum_{i=1}^n II(e_i, e_i), \xi\right) = -\sum_{i=1}^n \bar{g}(\overline{\nabla}_{e_i}\xi, e_i) \\ &= -\left\{\sum_{i=1}^n \bar{g}(\overline{\nabla}_{e_i}\xi, e_i) + \bar{g}(\overline{\nabla}_{e_{n+1}}\xi, e_{n+1})\right\} = -\operatorname{div}_{\overline{M}}\xi, \end{aligned} \tag{9}$$

where $\operatorname{div}_{\overline{M}}\xi$ denotes the divergence of $\xi$ on $\overline{M}$. Applying (2) to (9), after a tedious computation we find

$$nH = \rho\Delta\varphi + (\nabla\varphi)\rho + \frac{\rho}{f}(\nabla\varphi)f, \tag{10}$$

where $\nabla$ and $\Delta$ denote the gradient and the Laplacian in the metric of $M$ respectively, and

$$\rho = \frac{1}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}}.$$

By the hypothesis of the theorem, $f$ and $\varphi$ are radially symmetric, and $M$ is strongly symmetric, so it is easy to see that $\rho$ is also radially symmetric. Note that any geodesic $\gamma : [0, \infty) \to M$ parametrized by arclength with $\gamma(0) = o$ is a ray emanating from $o$ (see [9]). Let $\partial_r$ be the velocity vector of $\gamma$. Then $\partial_r$ is a smooth unit vector field on $M - \{o\}$. For any radially symmetric function $h = h(r)$ on $M$, we have $\nabla h = h'\partial_r$. Thus by (5) we have

$$nH = \rho\big[\varphi'' + (\Delta r)\varphi'\big] + \varphi'\rho' + \frac{\rho}{f}\varphi' f'. \tag{11}$$

Note that

$$\rho = \frac{1}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}} = \frac{f}{\sqrt{1 + (\varphi' f)^2}} \tag{12}$$

and that

$$\rho' = \frac{f' - f^3\varphi'\varphi''}{[1 + (\varphi' f)^2]^{\frac{3}{2}}}. \tag{13}$$

Applying (12) and (13) to (11), we obtain

$$nH = \frac{f[\varphi'' + (\Delta r)\varphi']}{\sqrt{1+(\varphi' f)^2}} + \varphi'\left\{\frac{f' - f^3\varphi'\varphi''}{[1+(\varphi' f)^2]^{\frac{3}{2}}}\right\} + \frac{\varphi' f'}{\sqrt{1+(\varphi' f)^2}}. \tag{14}$$

Since $\Sigma$ is minimal, we have

$$f\left[\varphi'' + (\Delta r)\varphi'\right]\left[1+\left(\varphi' f\right)^2\right] + f'\varphi' - f^3\left(\varphi'\right)^2\varphi'' + f'\varphi'\left[1+\left(\varphi' f\right)^2\right] = nH\left[1+\left(\varphi' f\right)^2\right]^{\frac{3}{2}} = 0,$$

that is,

$$f\varphi'' + f\varphi'\left[1+\left(\varphi' f\right)^2\right]\Delta r + f'\varphi'\left[2+\left(\varphi' f\right)^2\right] = 0.$$

Multiply the two sides of the above equality by $\varphi'$, we find

$$\begin{aligned} f\varphi'\varphi'' &= -f\left[\varphi'\right]^2\left[1+\left(\varphi' f\right)^2\right]\Delta r - f'\left[\varphi'\right]^2\left[2+\left(\varphi' f\right)^2\right] \\ &\leqslant -\left[\varphi'\right]^2\left[1+\left(\varphi' f\right)^2\right]\left(f\Delta r + f'\right). \end{aligned} \tag{15}$$

It is a known fact that

$$\Delta r = \frac{V'(r)}{V(r)} = \left[\log V(r)\right]' \tag{16}$$

(see [9,17]). By (15), (16), if $V(r)f(r)$ is increasing on $[0, R)$, then we have

$$\begin{aligned} \varphi'\varphi'' &\leqslant -\left[\varphi'\right]^2\left[1+\left(\varphi' f\right)^2\right]\left\{\Delta r + \left[\log f(r)\right]'\right\} \\ &\leqslant -\left[\varphi'\right]^2\left[1+\left(\varphi' f\right)^2\right]\left\{\log\left[V(r)f(r)\right]\right\}' \leqslant 0. \end{aligned}$$

This implies that

$$\left[\left(\varphi'\right)^2\right]' \leqslant 0.$$

This means that $(\varphi')^2$ is decreasing on $[0, R)$. But $\varphi'(0) = 0$, so we have $\varphi' \equiv 0$, and hence $\varphi$ is constant on $\Omega$. This proves the part (1) of the theorem.

By a results of [17], we know that if $M$ is strongly symmetric around $o$ with $\dim M = n$, then

$$\lim_{r\to 0} r\Delta r = n - 1. \tag{17}$$

Then there exists $R_0 \in (0, R)$ such that $\Delta r > 0$ on $B(o, R_0)$, and hence $V(r)$ is increasing on $[0, R_0)$. Thus, by the part (1) of the theorem, $\varphi$ is constant on $B(o, R_0)$. This proves the part (2) of the theorem.

If $M$ is a Cartan–Hadamard manifold, by the Laplacian comparison theorem (see [9]) (comparing with the $n$-dimensional Euclidean space $\mathbb{R}^n$), we see that

$$\Delta r \geqslant \Delta_0 r = \frac{n-1}{r} > 0$$

for $r > 0$, where $\Delta_0$ denotes the Laplacian of $\mathbb{R}^n$. Then $V(r)$ is increasing on $[0, +\infty)$, hence, again use the part (1) of the theorem, $\varphi$ is constant on $\Omega$. The part (3) of the theorem is proved, and the proof of the theorem is finished. □

## 5. Totally geodesic submanifolds in $M \times_f \mathbb{R}$

In this section, we study totally geodesic submanifolds in the warped product $\overline{M} = M \times_f \mathbb{R}$, where $M$ is an $n$-dimensional Riemannian manifold.

**Lemma 5.1.** *Let $N$ be a connected totally geodesic submanifold of the warped product $\overline{M} = M \times_f \mathbb{R}$ with $\dim N < n$. Let $h$ be the height function of $N$, that is, $h = \sigma|_N$. Then either $h$ has no extreme points on $N$, or $N$ is a part of a leave of the warped product $\overline{M}$.*

**Proof.** If $h$ has an extreme point $p \in N$, the $N$ is tangent to the leave $L_{h(p)} \equiv M \times \{h(p)\}$. But we know that all leaves are totally geodesic hypersurfaces of $\overline{M}$, thus $N \subset L_{h(p)}$. □

**Theorem 5.2.** *Consider the warped product $\overline{M} = M \times_f \mathbb{R}$. If $M$ has no $k$-dimensional compact totally geodesic submanifolds $(1 \leqslant k \leqslant n)$, neither has $\overline{M}$. In particular, if $M$ has no closed geodesics, neither has $\overline{M}$.*

**Proof.** Let $N$ be a $k$-dimensional connected compact totally geodesic submanifold of $\overline{M}$, then the height function $h$ of $N$ has extreme points. By Lemma 5.1 we know that $N$ is a part of a leave $L$ of $\overline{M}$, and hence $N$ is a totally geodesic submanifold of $L$. Since $\pi|_L : L \to M$ is an isometric, then it is obvious that $\pi(N)$ is a totally geodesic submanifold of $M$. This contradicts the hypothesis of the theorem. □

**Theorem 5.3.** *Let $\overline{M} = M \times_f \mathbb{R}$. Suppose that $M$ is complete and that $M$ has no compact totally geodesic submanifolds. Then any complete totally geodesic submanifold of $\overline{M}$ is not bounded.*

**Proof.** Since both $M$ and $\mathbb{R}$ are complete, the warped product $\overline{M}$ is also complete (see [14]). Let $N$ be a complete totally geodesic submanifold of $\overline{M}$. If $N$ is bounded, then the closure of $N$ is compact since $\overline{M}$ is complete. Then the completeness of $N$ implies that $N$ is compact. Thus $N$ is compact totally geodesic submanifold of $\overline{M}$. This contradicts the result of Theorem 5.2. So $N$ is not bounded. □

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